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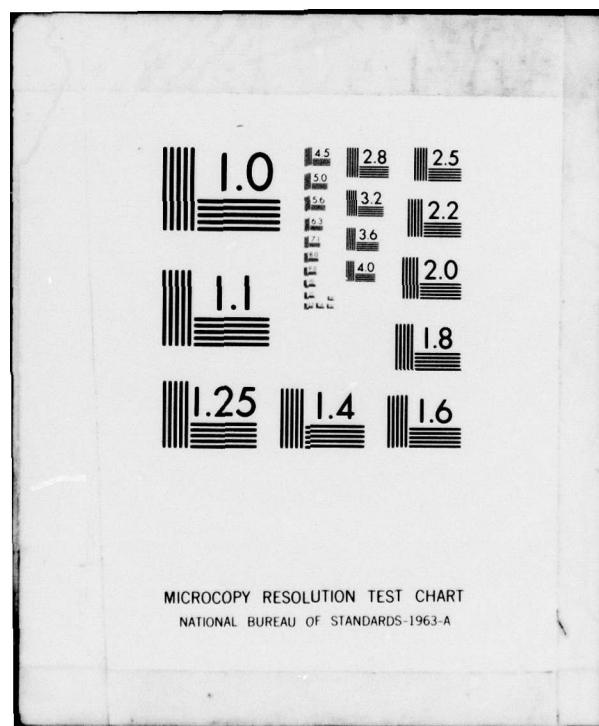
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HOGAN'S PIES EXAMPLE AND
LEMKE'S ALGORITHM

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ABSTRACT

Newton's method for generalized equations (Josephy [3]) ~~has been~~ applied to the economic equilibrium problem of the Project Independence Evaluation System (PIES) Energy Model. (Josephy [4]) The resulting algorithm involves solving a sequence of linear complementarity problems. Lemke's complementary pivot algorithm is used for this purpose. In this paper, ~~we show~~ that the linear complementarity problems will be copositive plus when the negative of the elasticity matrix, $-e$, of the consumer's quantity vs. price relation has the following properties: (1) positive diagonals, (2) negative off-diagonals, and (3) strict diagonal dominance. These conditions are satisfied for Hogan's example. (Hogan [2]) Thus, Lemke's algorithm will either converge to a solution or show that no solution exists. Under the conditions of Josephy [3], Theorem 1 of a solution to the linear complementarity problems will always exist. Hence, Lemke's algorithm can be used when the conditions of the Theorem 1 of Josephy [3] are satisfied.

AMS(MOS) Subject Classification: 90A15, 90C99, 49D99

Key Words: PIES Energy Model, Complementarity, Economic Equilibrium, Lemke's Complementary Pivot Algorithm, Copositive Plus

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Significance and Explanation

Newton's method for generalized equations (Josephy [3]) has been used to compute solutions to an economic equilibrium problem found in Hogan [2] (Josephy [4]). This economic equilibrium problem is a simplified version of the Project Independence Evaluation System (PIES) Energy Model developed by the Federal Energy Administration. The computational algorithm used at each iteration of Newton's method in Josephy [4] is Lemke's complementary pivot algorithm. In this paper, we give the analytical justification for the success of Lemke's algorithm in the computation of the economic equilibrium for Hogan's problem.

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HOGAN'S PIES EXAMPLE AND LEMKE'S ALGORITHM

Norman H. Josephy

1. INTRODUCTION

Newton's method for generalized equations (Josephy [3]) has been applied to the problem of computing the economic equilibrium of the Project Independence Evaluation System (PIES) Energy Model (Josephy [4]). In this paper, we show that the linear complementarity problems arising when Newton's method for generalized equations is applied to Hogan's PIES example (Hogan [2]) are copositive plus.

We recall in section 2 the structure of a competitive market equilibrium. Hogan's example is described in section 3. The application of Newton's method for generalized equation to Hogan's example is given in section 4, where the resulting linear complementarity problems are shown to be copositive plus.

2. EQUILIBRIUM IN A MARKET ECONOMY

The competitive market which underlies the PIES model consists of two classes of agents, suppliers and consumers, and two classes of goods, factors of production and consumable goods. The suppliers, faced with a perceived demand for consumable goods, convert factors of production into consumable goods, sustaining a cost for conversion to and delivery of consumable goods, and charging a price for those goods. The consumers purchase consumable goods at levels dependent upon the prices of all goods.

Suppose there are n consumable goods. Let $q \in \mathbb{R}_+^n$ and $p \in \mathbb{R}_+^n$, where q_i is the consumption level of the i th good, and p_i is the price of the i th good. For a perceived demand q of consumable goods, the supplier will charge prices no lower than $p_S(q) \in \mathbb{R}_+^n$. The consumer, desiring level q of consumable goods, is willing to pay no more than prices $p_D(q) \in \mathbb{R}_+^n$. A market equilibrium is a quantity vector q and a price vector p such that $p = p_D(q) = p_S(q)$. That is, at price p , the market for all goods will clear. The producers will supply at price p a level of consumable goods q which the consumers are willing to purchase at price p . The supply relation p_S is typically determined by the solution of a cost minimization problem modeling the engineering/technological processes involved in conversion of factors of production to delivered consumable goods. The demand relation p_D is traditionally a behavioral model econometrically determined from historical data.

For the case of n consumable goods, the equilibrium quantity q^* and price p^* and factor levels x^* for the competitive market with supply modeled as a linear program

minimize $\langle c, x \rangle$	total cost of production
$Ax \geq q$	demand requirements
$Bx \geq b$	non-demand constraints
$x \geq 0$	non-negative factor levels

and a log-linear consumer demand model

$$\ln(q_i/q_i^0) = \sum_{j=1}^n e_{ij} \ln(p_j/p_j^0), \quad i=1, \dots, n$$

where e is the elasticity matrix and q^0, p^0 are fixed reference values of p_D , that is, $p_D(q^0) = p^0$, is a triple $(q^*, p^*, x^*) \geq 0$ satisfying the following equilibrium conditions.

(EQ. 1) x^* solves: minimize $\langle c, x \rangle$ subject to
 $Ax \geq q^*, Bx \geq b, x \geq 0$.

(EQ. 2) For some multiplier s^* , (p^*, s^*) solve the dual problem: maximize $\langle q^*, p \rangle + \langle b, s \rangle$ subject to
 $A^T p + B^T s \leq c, (p, s) \geq 0$.

(EQ. 3) $\ln(q_i^*/q_i^0) = \sum_{j=1}^n e_{ij} \ln(p_j^*/p_j^0), i=1, \dots, n$.

3. THE HOGAN PIES EXAMPLE

As an illustration of PIES, Hogan [2] described a simplified situation which included the major aspects of the PIES model. Figure 1 illustrates the structure of Hogan's example. The factors of production are coal, crude oil, steel and capital. The consumable goods are coal, light oil and heavy oil. There are two coal mining regions, two crude oil drilling regions, two oil refineries, and two demand regions. Coal can be mined in region i at level c_{ij} , $j = 1, 3$, at three different levels j for differing production costs. Oil can be drilled in region i at level o_{ij} , $j = 1, 2$, at two different levels j for differing production costs.

Coal is transported at level c_{Tij} from coal region i to demand region j . Crude oil is transported from oil region i to refinery j in a quantity o_{Tij} , where it is refined into a fixed proportion of

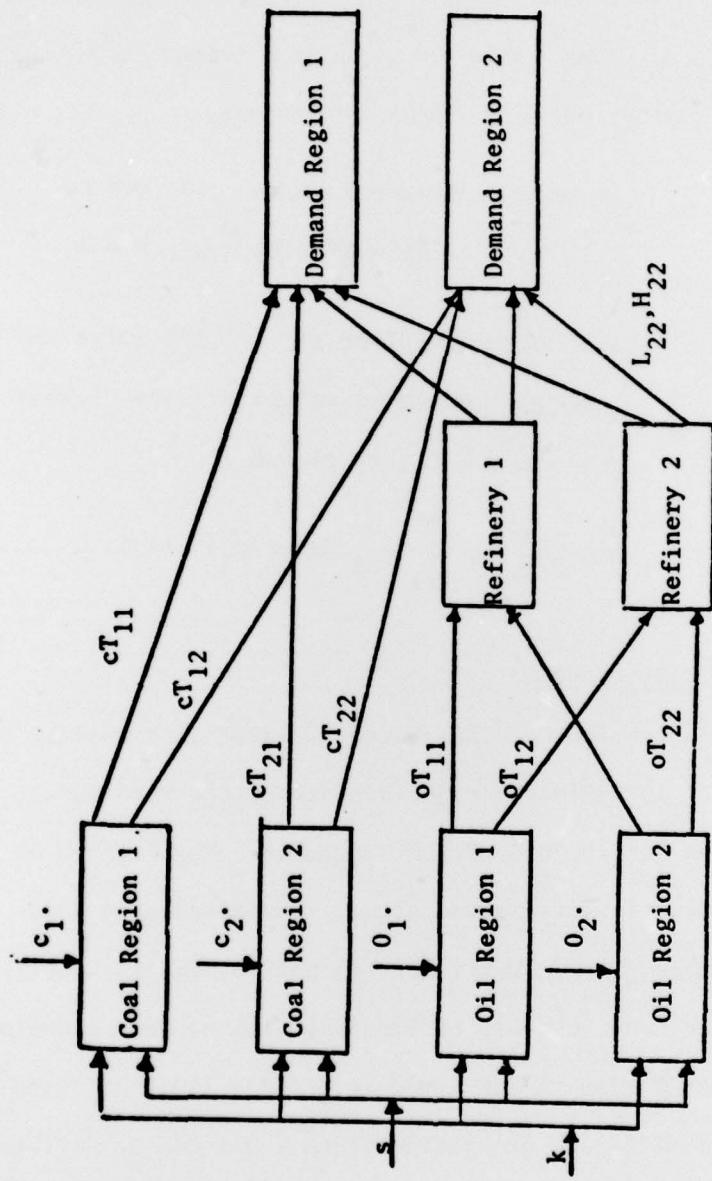


Figure 1. The Hogan PIES Example.

light and heavy oil. Refinery 1 converts 60% of its crude oil into light oil, while refinery 2 converts 50% of its crude oil into light oil. Light oil L_{ij} and heavy oil H_{ij} are transported from refinery i to demand region j .

The linear program representing the minimum cost of production and delivery of consumable goods to meet a demand $q \in \mathbb{R}_+^6$ is of the form

$\min \langle r, x \rangle$ subject to

$Ax \geq q$ demand requirements

$Bx = 0$ transportation flow balance

$Ex \leq a$ factor of production upper bounds

$Fx \leq b$ steel and capital resource constraint

$x \geq 0$ non-negative factors of production

The components of x represent the levels of production of coal and crude oil, levels of coal transported to demand regions, levels of crude oil transported to refineries, and levels of refined oil transported to the demand regions. The coefficient r_i is the unit cost of the activity associated with x_i . The vector Ax consists of levels of consumable goods delivered to the demand regions. Bx represents the conservation of materials transported through a region or refinery. Fx is the vector of steel and capital consumed in the processing of factors x .

The demand relationship between the desired quantity of a consumable good in a particular demand region, q_i , and the prices consumers are willing to pay for such goods, $\{p_j\} j=1, \dots, 6$, is given by

$$\ln(q_i/q_i^0) = \sum_{j=1}^6 e_{ij} \ln(p_j/p_j^0), \quad i=1, \dots, 6$$

4. NEWTON'S METHOD AND LEMKE'S ALGORITHM

An equilibrium problem which is equivalent to that of Hogan's example when consumable good prices and quantities are positive is the following. The supplier solves the linear program

$$\min \langle c, x \rangle, \text{ subject to } \hat{A}x \geq q, \hat{B}x \geq b, \text{ and } x \geq 0.$$

Note that the original form of Hogan's example contains transportation flow equations $Bx=0$. However, due to the flow conservation nature of the constraints, the matrix B factors into the form $B = (I, \bar{B})$.

Hence, replacing $0=Bx$ with $-\bar{B}x_2 \geq 0$, where $x = (x_1, x_2)$ and $x_1 = -\bar{B}x_2$, and similarly decomposing the other inequalities and replacing x_1 with $-\bar{B}x_2$, one obtains a linear program with no equality constraints and fewer variables. The matrix \hat{B} represents all non-demand constraints, while \hat{A} represents the demand constraints. The subscript 2 on x has been dropped. The supply equals demand equilibrium condition is replaced by the following conditions:

$$p - p_D(q) \geq 0, \quad q \geq 0, \quad \langle q, p - p_D(q) \rangle = 0.$$

These conditions are that no supply price p_i is ever below the corresponding demand price $p_D(q)_i$, and if a positive quantity of q_i is demanded, then $p_i = p_D(q)_i$. The equilibrium conditions for this problem are, dropping hats on A , B and c ,

$$(1) \quad c - A^T p - B^T s \geq 0, \quad x \geq 0, \quad \langle x, c - A^T p - B^T s \rangle = 0;$$

$$(2) \quad Ax - q \geq 0, \quad p \geq 0, \quad \langle p, Ax - q \rangle = 0;$$

$$(3) \quad Bx - b \geq 0, \quad s \geq 0, \quad \langle s, Bx - b \rangle = 0;$$

$$(4) \quad p - p_D(q) \geq 0, \quad q \geq 0, \quad \langle q, p - p_D(q) \rangle = 0.$$

We can now write equilibrium conditions (1) - (4) as a generalized equation. (See Josephy [3]). Let $K: = \mathbb{R}_+^\ell$, where $\ell: = m+n+v+n$, and $x \in \mathbb{R}^m$, $p \in \mathbb{R}^n$, $s \in \mathbb{R}^v$ and $q \in \mathbb{R}^n$. Define F by

$$F(z): = \begin{bmatrix} c \\ 0 \\ -b \\ -p_D(q) \end{bmatrix} + \begin{bmatrix} 0 & -A^T & -B^T & 0 \\ A & 0 & 0 & -I \\ B & 0 & 0 & 0 \\ 0 & I & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ p \\ s \\ q \end{bmatrix}$$

where $z: = (x, p, s, q)$. Then finding an economic equilibrium is equivalent to finding a solution to the generalized equation

$$(5) \quad 0 \in F(z) + N_K(z).$$

The linearization of (5) at a point $\bar{z}: = (\bar{x}, \bar{p}, \bar{s}, \bar{q})$ is given by

$$(6) \quad 0 \in LF_{\bar{z}}(z) + N_K(z),$$

where

$$LF_{\bar{z}}(z) = \begin{bmatrix} c \\ 0 \\ -b \\ -p_D(\bar{q}) + p'_D(\bar{q})\bar{q} \end{bmatrix} + \begin{bmatrix} 0 & -A^T & -B^T & 0 \\ A & 0 & 0 & -I \\ B & 0 & 0 & 0 \\ 0 & I & 0 & -p'_D(\bar{q}) \end{bmatrix} z.$$

The linear complementarity problem corresponding to the linearization (6) about \bar{z} is

$$LF_{\bar{z}}(z) \geq 0, z \geq 0, \langle z, LF_{\bar{z}}(z) \rangle = 0.$$

Lemke's algorithm (see Cottle and Dantzig [1]) can be applied to this linear complementarity problem if the matrix

$$M := \begin{bmatrix} 0 & -A^T & -B^T & 0 \\ A & 0 & 0 & -I \\ B & 0 & 0 & 0 \\ 0 & I & 0 & -p_D^T(\bar{q}) \end{bmatrix}$$

is copositive plus, where copositive plus is defined as follows:

Definition 1. A matrix M is copositive plus if and only if

- (1) $\langle u, Mu \rangle \geq 0$ for all $u \geq 0$, and
- (2) if $u \geq 0$ and $\langle u, Mu \rangle = 0$, then $(M+M^T)u = 0$

See Cottle and Dantzig (1) for further details. Since M is copositive plus if and only if $-p_D^T(\bar{q})$ is copositive plus, the applicability of Lemke's algorithm rests on showing that $-p_D^T(\bar{q})$ is copositive plus. Recall the demand relations

$$\ln(q_i/p_i^0) = \sum_j \epsilon_{ij} \ln(p_j/p_j^0), \quad i=1, \dots, 6.$$

Inverting these relations yield

$$\ln(p_j/p_j^0) = \sum_k \alpha_{jk} \ln(q_k/q_k^0), \quad i=1, \dots, 6,$$

where $(\alpha) = (\epsilon)^{-1}$. The elasticity matrix ϵ is of the form

$$\epsilon = \begin{bmatrix} \epsilon_{11} & 0 & \epsilon_{12} & 0 & \epsilon_{13} & 0 \\ 0 & \epsilon_{11} & 0 & \epsilon_{12} & 0 & \epsilon_{13} \\ \epsilon_{21} & 0 & \epsilon_{22} & 0 & \epsilon_{23} & 0 \\ 0 & \epsilon_{21} & 0 & \epsilon_{22} & 0 & \epsilon_{23} \\ \epsilon_{31} & 0 & \epsilon_{32} & 0 & \epsilon_{33} & 0 \\ 0 & \epsilon_{31} & 0 & \epsilon_{32} & 0 & \epsilon_{33} \end{bmatrix}$$

where $E: = (e_{ij})$ are the elasticities of quantity versus price for coal, light oil and heavy oil. The zero elements in ϵ correspond to the independence of quantities demanded in one region and prices in the other region. If we define $F = (f_{ij}) = E^{-1}$, then it is easily seen that

$$\alpha := \epsilon^{-1} = \begin{bmatrix} f_{11} & 0 & f_{12} & 0 & f_{13} & 0 \\ 0 & f_{11} & 0 & f_{12} & 0 & f_{13} \\ f_{21} & 0 & f_{22} & 0 & f_{23} & 0 \\ 0 & f_{21} & 0 & f_{22} & 0 & f_{23} \\ f_{31} & 0 & f_{32} & 0 & f_{33} & 0 \\ 0 & f_{31} & 0 & f_{32} & 0 & f_{33} \end{bmatrix}$$

The matrix $-E$ has the following properties for Hogan's example:

- (1) positive diagonals ,
- (2) negative off-diagonals ,
- (3) strict diagonal dominance ,

where (Plemmons, [5]) a matrix M is strictly diagonally dominant if and only if there exist positive constants $\{d_i\}$ such that for each i ,

$$m_{ii}d_i > \sum_{j \neq i} |m_{ij}| \cdot d_j .$$

For the Hogan example, the d_i can be chosen to be 1. It follows from Plemmons (5) that $-F$ is strictly positive. The matrix $-p_D'(\bar{q})$ is

$$(-p_D'(\bar{q}))_{ij} = \frac{-\bar{p}_i}{\bar{q}_j} \alpha_{ij} .$$

Thus, for positive \bar{p} and \bar{q} , $-p_D'(\bar{q})$ is non-negative, and hence Condition 1 of the definition of a copositive plus matrix is satisfied. To show that Condition 2 also holds, it will be convenient to define

a matrix $G = -p_D'(\bar{q})$, where $G_{ij} = \frac{-\bar{p}_i}{\bar{q}_j} \alpha_{ij}$. Thus, G has the form

$$G = \begin{bmatrix} g_{11} & 0 & g_{12} & 0 & g_{13} & 0 \\ 0 & g_{21} & 0 & g_{22} & 0 & g_{23} \\ g_{31} & 0 & g_{32} & 0 & g_{33} & 0 \\ 0 & g_{41} & 0 & g_{42} & 0 & g_{43} \\ g_{51} & 0 & g_{52} & 0 & g_{53} & 0 \\ 0 & g_{61} & 0 & g_{62} & 0 & g_{63} \end{bmatrix} ,$$

where all $g_{ij} > 0$, $i=1, \dots, 6$, $j=1, \dots, 3$.

To show the validity of Condition 2 of Definition 1, suppose $u \geq 0$ and $\langle u, Gu \rangle = 0$. Let $w: = (u_1, u_3, u_5)$ and $z: = (u_2, u_4, u_6)$.

Let g_i denote the row (g_{i1}, g_{i2}, g_{i3}) , and $g_i \cdot w = g_{i1}w_1 + g_{i2}w_2 + g_{i3}w_3$, for $i=1, \dots, 6$. Then $Gu = (g_1 \cdot w, g_2 \cdot z, g_3 \cdot w, g_4 \cdot z, g_5 \cdot w, g_6 \cdot z)$ and $\langle u, Gu \rangle = u_1 \cdot (g_1 \cdot w) + u_2 \cdot (g_2 \cdot z) + u_3 \cdot (g_3 \cdot w) + u_4 \cdot (g_4 \cdot z) + u_5 \cdot (g_5 \cdot w) + u_6 \cdot (g_6 \cdot z)$. Suppose some $u_i \neq 0$, say u_1 . Then $u_1 > 0$ and thus $g_{11}u_1 > 0$; hence $g_1 \cdot w > 0$ and thus $u_1 \cdot (g_1 \cdot w) > 0$, contradicting $0 = \langle u, Gu \rangle$. A similar argument works for any choice of $u_i > 0$. Hence, if $0 = \langle u, Gu \rangle$, then $u = 0$. Thus, Condition 2 holds, and $-p_D'(q)$ is copositive plus. Indeed, the above analysis actually shows that $-p_D'(q)$ is strictly copositive. (See Cottle and Dantzig (1) for details on strict copositivity).

An interesting property of $-p_D'(q)$ relates strict diagonal dominance of $-p_D'(q)$ to the same property for $-\alpha$. Note that

$$-p_D'(q)_{ij} = \frac{-p_i}{q_j} \alpha_{ij} .$$

For all $q_j > 0$, take $d_j = q_j$ in the definition given for strict diagonal dominance. Then $-p_D'(q)$ is strictly diagonally dominant if and only if, for each i ,

$$\frac{-p_i}{q_j} \alpha_{ii} d_i > \sum_{j \neq i} \frac{p_i}{q_j} |\alpha_{ij}| d_j ,$$

and by replacing d_i with q_i , we have

$$-p_i \alpha_{ii} > p_i \cdot \sum_{j \neq i} |\alpha_{ij}| .$$

For $p_i > 0$, this reduces to

$$- \alpha_{ii} > \sum_{j \neq i} |\alpha_{ij}| .$$

Thus, $-p'_D(q)$ will be strictly diagonally dominant when α is strictly diagonally dominant. The implications of strict diagonal dominance are discussed in great detail in Plemmons (5).

A final remark is in order concerning the assumption of positive prices and quantities of consumable goods. The form of the function p_D assumed by Hogan is undefined when any quantity q_i is zero, since the diagonal elements of α are negative. Thus, an equilibrium solution which satisfies the condition that the supplier's price must not be less than the consumer's price must have $q > 0$ and (since $p_D(q) > 0$) $p \geq p_D(q) > 0$. Thus, it follows from Josephy [3, Theorem 1] that in a sufficiently small neighborhood of the equilibrium price and quantity, the linearized problem will yield positive p and q .

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